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Weakly nonlinear internal wave fronts trapped in contractions

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The propagation of weakly nonlinear, long internal wave fronts in a contraction is considered in the transcritical limit as a model for the establishment of virtual controls. It is argued that the appropriate equation to describe this process is a variable coefficient Korteweg–de Vries equation. The solutions of this equation are then considered for compressive and rarefaction fronts. Rarefaction fronts exhibit both normal and virtual control solutions. However, the interaction of compressive fronts with contractions is intrinsically unsteady. Here the dynamics take two forms, interactions with the bulk of the front and interactions with individual solitary waves separating off from a front trapped downstream of the contraction.

1. Introduction

The concept of a virtual control was introduced by Wood (1968) in studying the hydraulic, or hydrostatic, limit of the steady selective withdrawal of a stratified fluid through a contraction. In contrast to normal controls, which only occur at the maximum topographic perturbation to the flow, virtual controls can occur anywhere along the contraction and are associated with the behaviour of a particular linear long-wave mode changing from subcritical to supercritical. Wood showed that for a continuously stratified fluid a self-similar solution exists which accelerates the flow from a stagnant reservoir through an infinite number of virtual controls. This solution was studied further by Benjamin (1981), who demonstrated the hydraulically controlled nature of the flow, and by Armi & Williams (1993), who, still in the hydraulic limit, considered both experimentally and theoretically deviations from this flow. An excellent example of a virtual control can be seen in figure 6 of Armi (1986), which shows the flow of a two-layer fluid through a contraction. Here the interface remains flat past the throat of the contraction and then, from a point downstream of the throat, slowly increases in height. Thus the flow must pass through two virtual controls, one on the upstream side of the throat which waves propagating from downstream cannot reach and one on the downstream side at which point waves become trapped.

As demonstrated by Armi (1986) and Killworth (1992) there is a fundamental difference between the hydraulic flow of a stratified fluid through a contraction and over a sill. In the former both virtual and normal controls can occur, whereas in the latter only normal controls occur. The process of the generation of normal controls for the flow of a stratified fluid over topography is well understood from weakly nonlinear, long-wave theory (e.g. see Grimshaw & Smyth 1986). Here transcritical flow over a sill creates resonantly forced waves which generate subcritical conditions upstream and supercritical conditions downstream. Clarke & Grimshaw (1994) showed that, in general, in the weakly nonlinear, long-wave limit transcritical flow through a contraction and over topography are equivalent, so long as in the former case either the perturbation to the width of the channel varies with depth or the oncoming fluid has a vertically varying velocity. In these circumstances a vertical velocity is induced resulting in the deflection of isopycnals and the consequent generation of resonant waves. Moreover, if the horizontal velocity is not vertically sheared and the channel cross-section is independent of depth, Clarke & Grimshaw (1994) showed that a similar situation may still occur due to non-Boussinesq effects. However, this situation only occurs in a very narrow transcritical velocity band with relatively small amplitude waves, and still with only normal control type solutions. This contrasts with hydraulic theory which demonstrates that virtual and normal controls may occur even in this joint limit. Thus, another mechanism must cause the generation of hydraulic control in these circumstances.

Since virtual controls are fundamental to the problem of the selective withdrawal of a stratified fluid through a contraction, we consider the possibility that a mechanism for their generation may be found by considering the initial-value problem of selective withdrawal in the presence of a weak contraction. Indeed, Kao (1976) presented the results of unsteady experiments confirming that selective withdrawal does not occur upstream of a contraction if the linear long-wave speed of the first vertical mode vanished in the contraction, i.e. the first mode experienced a virtual control. The unsteady theory of selective withdrawal (e.g. see Pao & Kao 1974; Imberger & Patterson 1990; Clarke & Imberger 1995) demonstrates that columnar disturbances or shear fronts may be responsible for the generation of steady selective withdrawal flows. These are horizontally propagating waves of zero frequency across which the velocity and density have differing upstream and downstream limits. Internal wave fronts, shear fronts or columnar disturbances, referred to here collectively as fronts, do not just occur in selective withdrawal. For example, the theory of Pao & Kao (1974) was preceded by the work of McEwan & Baines (1974) who considered the establishment of an experimental shear flow. The waves found to establish this flow were identical to those found in the establishment of selective withdrawal. Further, it would be expected that in the experiments of Armi (1986) an interfacial front would be generated downstream of the contraction when the flow commences. The transcritical interaction of these fronts with contractions would appear to be a possible mechanism for the creation of virtual controls. It is this unsteady interaction which we study here.

To study the unsteady interactions of fronts with contractions we must in general separate out the near-critical mode. This assumption then allows analytical progress to be made using weakly nonlinear, long-wave theory. In the problem of selective withdrawal this neglect of other modes is not a dramatic restriction, since it is the effect of the contraction on the first mode which is fundamental to determining the behaviour upstream of the contraction. For other manifestations of fronts it would be expected that there is also a fundamental mode which largely determines the behaviour of the system. For two-layer flow there is only one upstream propagating mode, whereas in the work of McEwan & Baines (1974) again the first mode is the fundamental mode. It is the transcritical behaviour of these fundamental modes which is of interest.

In § 2, following Clarke & Grimshaw (1994), the equations describing the transcritical propagation of weakly nonlinear, long internal waves through a weak contraction are presented. Consideration of these equations for the initial condition of a front then leads to a model equation, a variable coefficient Korteweg–de Vries (vKdV) equation, for the study of the interaction of fronts with contractions and consequently the establishment of virtual controls. The hydraulic limit of the vKdV equation is considered in § 3. Fronts have two possible forms, either rarefactions or undular bores, corresponding to negative and positive amplitudes. The respective behaviour of each of the two forms is presented in §§ 4 and 5. The results are summarized and related to previous work on virtual controls in § 6.

2. Problem formulation

Consider the inviscid, non-diffusive flow of a stratified fluid in a duct. A Cartesian coordinate system $h(x, y, z)$ is introduced, where h is the undisturbed height of the free-surface above the origin, x is the horizontal coordinate along the duct, y is the transverse coordinate and z is the vertical coordinate, being positive upwards. The density is $\rho_0\rho(z-\zeta)$, where ζ is the non-dimensional vertical particle displacement. The Boussinesq parameter, which characterizes the strength of the density stratification, is then defined as $\beta = \rho(0) - \rho(1)$. In turn the reduced gravity is $g' = \beta g$, where g is the magnitude of the acceleration due to gravity. The normalized buoyancy frequency is then

$$N^2 = -\frac{\rho'}{\beta\rho}. \tag{2.1}$$

Time is $(h/g')^{1/2}t$, the fluid velocities are $(g'h)^{1/2}(u, v, w)$ and the free surface displacement is $eh\eta$, where the binary parameter e is 0 for a rigid lid boundary condition and 1 for a free surface condition. Finally the pressure is $\rho_0 gh(P(z) + \beta p)$, where $P' = -\rho(z)$.

The waves in the channel are assumed to have amplitude $O(\alpha)$ and wavelength $O(\mu^{-1})$. It is assumed that this is also the lengthscale of the perturbation of the contraction, which has amplitude $O(\epsilon)$, and that

$$\alpha, \epsilon, \mu \ll 1. \tag{2.2}$$

Now assuming the usual balance for long, weakly nonlinear waves,

$$\alpha = \mu^2, \tag{2.3}$$

we introduce

$$(x, y, t) = \alpha^{-1/2}(x', y', t'), \tag{2.4}$$

and

$$b'_{\pm} = \pm b'_0(1 + \epsilon f_{\pm}(x', z)), \tag{2.5}$$

where $y = b'_{\pm}$ denotes the transverse boundaries of the channel. The mean perturbation to the channel width is

$$f = \tfrac{1}{2}(f_+ + f_-). \tag{2.6}$$

A mean flow $\bar{u}$ exists in the channel, which can be assumed to be independent of x and y. At this point no other restrictions are placed on $f_{\pm}$ and $\bar{u}$. The appropriate scaling is then

$$u = \bar{u} + \alpha u', \quad v = \epsilon v', \quad w = \alpha^{3/2} w', \quad p = \alpha p', \quad \zeta = \alpha\zeta', \quad \eta = \alpha\beta\eta'. \tag{2.7}$$

Hereinafter only the scaled variables will be used, and so the prime superscripts are omitted.

To investigate the behaviour of fronts trapped in a contraction it must be assumed that the amplitude of the transcritical front is significantly larger than that of all other modes. In general this would be expected to be the case as in this weakly nonlinear, long-wave limit all other modes will propagate away from the contraction. Further, the leading order dynamical effects of nonlinearity and dispersion are assumed to be of the same order of magnitude as the leading order effect of the contraction. These assumptions then determine the magnitude of α. As shown by Clarke & Grimshaw (1994) for general stratifications there are two possible choices for α. A full derivation of the following equations can be found therein.

Case 1 In the general case $\bar{u}$ and f vary with height. Let

$$\alpha = \epsilon^{1/2}, \tag{2.8}$$

with no restriction on β. The basic state, $\bar{u}(z)$, $\rho(z)$, is assumed to be close to a long-wave resonance, where

$$\bar{u} = -c(z) + \epsilon^{1/2}\Delta + O(\epsilon), \tag{2.9}$$

and c is such that there is a regular long-wave modal function $\phi(z)$, defined by

$$(\rho c^2 \phi_z)_z + \rho N^2 \phi = 0 \tag{2.10a}$$

with

$$\phi = \begin{cases} e\beta c^2 \phi_z & \text{on } z = 1 \\ 0 & \text{on } z = 0. \end{cases} \tag{2.10b}$$

The detuning parameter Δ quantifies the deviation of the system from resonance. A long-time scale is introduced:

$$\tau = \epsilon^{1/2} t, \tag{2.11}$$

and a perturbation solution for ζ is sought in the form

$$\zeta = A(x,\tau)\phi(z) + \epsilon^{1/2}\zeta^{(1)} + O(\epsilon). \tag{2.12}$$

Then it can be shown that A satisfies the forced KdV equation

$$A_\tau + \Delta A_x + rAA_x + sA_{xxx} = G_x, \tag{2.13a}$$

where

$$I = 2\int_0^1 \rho c^2 \phi_z^2 \, \mathrm{d}z, \tag{2.13b}$$

$$Ir = 3\int_0^1 \rho c^3 \phi_z^3 \, \mathrm{d}z, \tag{2.13c}$$

$$Is = \int_0^1 \rho c^3 \phi^2 \, \mathrm{d}z, \tag{2.13d}$$

$$IG = \int_0^1 \rho c^2 f \phi_z \, \mathrm{d}z. \tag{2.13e}$$

In general, since $\beta \ll 1$ the Boussinesq limit $\beta \to 0$ may be used, as it will only marginally effect the coefficients of (2.13).

Case 2 In the joint limit $\beta \to 0$ and $\bar{u}$ and f independent of height the forcing term in (2.13) disappears. However, Clarke & Grimshaw (1994) demonstrated that then a similar situation may still occur due to non-Boussinesq effects. Now let

$\bar{u}_z = f_z = 0$ and

$$\alpha = \epsilon, \quad \beta = \sigma\epsilon, \tag{2.14}$$

where, for the moment, $\sigma = O(1)$. Again it is assumed that $\bar{u}$ is close to a long-wave resonance, however now

$$\bar{u} = -c + \epsilon\delta + O(\epsilon^2), \tag{2.15}$$

and c is now a constant, and such that for a linear long-wave modal function $\phi(z)$

$$c^2\phi_{zz} + N^2\phi = 0 \tag{2.16a}$$

with

$$\phi = 0 \text{ on } z = 0, 1. \tag{2.16b}$$

Then the width-averaged perturbation to the mean flow due to the contraction is $\hat{\delta}$, where it can be shown that

$$\hat{\delta}_x = \frac{1}{2b_0}\int_{-b_0}^{b_0} \delta_x \,\mathrm{d}y = cf_x. \tag{2.17}$$

Introducing

$$\tau = \epsilon t, \tag{2.18}$$

and

$$\zeta = A(x,\tau)\phi(z) + \epsilon\zeta^{(1)} + O(\epsilon^2), \tag{2.19}$$

it can then be shown that A satisfies

$$A_\tau + (\varDelta A)_x + rAA_x + sA_{xxx} = \gamma\varDelta_x, \tag{2.20a}$$

where

$$I = 2\int_0^1 \phi_z^2 \,\mathrm{d}z, \tag{2.20b}$$

$$I\varDelta = I\hat{\delta} + \sigma c^3(\tfrac{1}{2}[\phi_z^2]_0^1 - e[\phi_z^2]_{z=1}), \tag{2.20c}$$

$$Ir = 3c\int_0^1 \phi_z^3 \,\mathrm{d}z, \tag{2.20d}$$

$$Is = c\int_0^1 \phi^2 \,\mathrm{d}z, \tag{2.20e}$$

$$I\gamma = \sigma c^2(e[\phi_z]_{z=1} - [\phi_z]_0^1). \tag{2.20f}$$

The coefficient of the forcing term, γ, is $O(\sigma)$, and so the amplitude of any forced wave is actually $O((\beta\epsilon)^{1/2})$ rather than $O(\epsilon)$. Clearly, in the Boussinesq limit, $\sigma \to 0$, no forced wave is generated.

In the absence of incident fronts, for both (2.13) and (2.20) two types of resonant solution occur: strong resonant solutions which are characterized by quasi-steady supercritical conditions downstream of the contraction and weak resonant solutions which are characterized by quasi-steady lee wave trains downstream of the contraction. In the terminology of hydraulic theory both these types of solution can be classified as exhibiting normal control, that is the solution is controlled at the maximum constriction.

In the first instance the effect of a shear front of amplitude $\varLambda$ propagating towards the contraction can be considered by uniformly perturbing the amplitude by an amount Λ. Then, (2.13) is invariant under the transformation

$$A \to A + \Lambda, \qquad \Delta \to \Delta - r\Lambda, \tag{2.21}$$

whereas (2.20) is only invariant with the additional transformation

$$\gamma \to \gamma + \Lambda. \tag{2.22}$$

Thus by perturbing the amplitude in (2.13) the effect is only to change the detuning parameter, leaving the forcing unchanged. The balance for a trapped front and a forced resonant solution are therefore identical. In (2.20) the forcing is also altered, which suggests that the balance between a trapped front and a resonant forced solution differ. In this latter case, the resonant forced solution is due to a balance between non-Boussinesq effects and the nonlinear terms, while the trapped front can occur solely due to a balance between the basic flow perturbation and the nonlinear terms.

For (2.13) Grimshaw & Smyth (1986) showed that the quasi-steady solutions form over a timescale $O(\epsilon^{-1/2})$. Thus it can be justifiably assumed that a quasi-steady solution has formed by the time any incident front has propagated to the vicinity of the contraction. If a strong resonant solution has formed the conditions are supercritical downstream of the contraction, and so, the front will be swept back downstream with the undular bores formed by the resonant solution. If a weak resonant solution forms then the front may be able to propagate upstream. Its only effect would then be to perturb the resonant solution, causing the detuning parameter to change, but leaving the forcing unchanged. Thus a front cannot be expected to alter the leading order balance which occurs for a controlled solution in this case. Moreover, the possibility of virtual controls occurring in this system can be dismissed by noting that (2.13) is also the model equation for flow over topography, in which case Killworth (1992) has shown that in general only normal controls can occur.

In the second case the effect of a front with amplitude $O(\epsilon)$ propagating towards the contraction would be expected to dominate any resonant waves, as typically $\epsilon \gg \beta$ or equivalently $|\Lambda| \gg \sigma$. It is this case that is our concern here. Consequently, we make the Boussinesq approximation, i.e. $\sigma \to 0$, whereupon (2.20) becomes

$$A_\tau + (\Delta A)_x + rAA_x + sA_{xxx} = 0. \tag{2.23}$$

This can be converted to canonical form by introducing

$$\tau^* = s\tau, \qquad \Delta^* = \frac{\Delta}{s}, \qquad A^* = \frac{rA}{6s}. \tag{2.24}$$

Then omitting the asterisks

$$A_\tau + (\Delta A)_x + 6AA_x + A_{xxx} = 0. \tag{2.25}$$

This vKdV equation forms the basis of our model for the interaction of fronts with contractions.

The front is assumed to be initially described by a step function, centred downstream of the contraction, where

$$A(x,0) = \Lambda S(x), \tag{2.26}$$

such that S decreases monotonically from 1 to 0 as x varies from $-\infty$ to ∞. The velocity perturbation in the contraction is assumed here to be of the form

$$\Delta = \Delta_0 - \Delta_1 \theta(x/x_a), \tag{2.27}$$

where $0 \leqslant \theta \leqslant 1$, with θ decreasing monotonically from 1 to 0 as $|x|$ varies from 0 to ∞. For example, all numerical simulations presented here use

$$\theta(x) = \mathrm{sech}^2 x. \tag{2.28}$$

In the limit $x_a \to 0$ the solutions depend only on the integral of θ over x. Further, in general, so long as both θ and S are smoothly varying the exact details of the two functions do not significantly effect the solutions. Note that Δ_0 represents the upstream velocity perturbation ($\Delta_0 \gtrless 0$ corresponds to subcritical/supercritical respectively), while $\Delta_1 > 0$ represents the magnitude of the contraction (see (2.17)).

Thus the problem to be investigated hereafter is described by (2.25), (2.26) and (2.27). For a given $\theta(x)$ this has four free parameters: Λ, Δ_0, Δ_1 and x_a. However, (2.25) is invariant under the transformation

$$\tau \to \lambda^{3/2}\tau, \quad x \to \lambda^{1/2}x, \quad \Delta \to \lambda^{-1}\Delta, \quad A \to \lambda^{-1}A, \tag{2.29}$$

where λ is an arbitrary (positive) parameter. Let

$$\lambda = (6|\Lambda|)^{-1}, \tag{2.30}$$

then it is apparent that the three important parameters for this problem are $|\Lambda|^{1/2}x_a$, $|\Lambda|^{-1}\Delta_0$ and $|\Lambda|^{-1}\Delta_1$. The terms long and short contractions refer hereinafter to the limits $|\Lambda|^{1/2}x_a \gg 1$ and $|\Lambda|^{1/2}x_a \ll 1$ respectively. Note that although the limit of short contractions is explicitly considered here, this is to determine how the long contraction limit is modified. In reality it would be expected that columnar vortices might form in the lee of short obstacles effectively modifying the flow field in the vicinity of the obstacle.

In the limit $\Delta_1 \to 0$ the interaction of fronts with contractions must be consistent with the solutions for $\Delta_1 = 0$, in which case (2.25) is simply the KdV equation. Approximate solutions for the evolution of a front are then well known (e.g. see Gurevich & Pitaevskii 1974 and Fornberg & Whitham 1978). Therefore, let the initial condition be a discrete step centred at $x = x_0 (< 0)$, such that

$$S(x) = 1 - H(x - x_0), \tag{2.31}$$

where $H(\cdot)$ is the Heaviside step function. Smoother initial conditions will add transients to the approximate solutions.

Based on the modulation theory of Whitham (1965), for $\Lambda > 0$ Gurevich & Pitaevskii (1974) showed that the solution is an undular bore which can be approximated by a slowly varying cnoidal wavetrain:

$$A = b + 2a\left\{\mathrm{cn}^2\kappa(x - x_0 - (U + \Delta_0)\tau) + \frac{1-m}{m} - \frac{E}{mK}\right\}, \tag{2.32a}$$

where

$$\kappa = \left(\frac{a}{m}\right)^{1/2}, \qquad U = 6\left(b + 2a\left(\frac{2-m}{3m} - \frac{E}{mK}\right)\right), \tag{2.32b}$$

m is the modulus of the Jacobian elliptic function cn, while $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind respectively. The parameters b, a and m are slowly varying functions of x and τ. The appropriate solution for an undular bore is

$$\left.\begin{aligned} b &= \varLambda\left(m-1+\frac{2E}{K}\right)\\ a &= \varLambda m\\ \frac{x-x_0}{\tau} &= \varDelta_0+2\varLambda\left(1+m-\frac{2m(1-m)K}{E-(1-m)K}\right)\end{aligned}\right\}\ \text{for } 0\leqslant m\leqslant 1, \tag{2.33}$$

which occupies the expansion fan

$$\varDelta_0-6\varLambda\leqslant\frac{x-x_0}{\tau}\leqslant\varDelta_0+4\varLambda. \tag{2.34}$$

At the leading edge as $m\to 1$ the solution approaches a solitary wave of amplitude $2\varLambda$, while at the trailing edge as $m\to 0$ the solution approaches linear waves propagating on a mean level $\varLambda$. In the limit $\varDelta_1\to 0$ this solution can only completely propagate to within the vicinity of the contraction if $4\varLambda\geqslant -\varDelta_0$. If this inequality is not satisfied an interaction will not occur. Conversely, if $6\varLambda<\varDelta_0$ the whole undular bore can propagate through the contraction, and it would be expected that the interaction between the undular bore and contraction will be rapid and the undular bore will propagate upstream, leaving a subcritical solution in the vicinity of the contraction.

For $\varLambda<0$ Fornberg & Whitham (1978) showed that asymptotically the dispersive term could be neglected. In this limit the solution is a rarefaction front of the form

$$A=\begin{cases}\varLambda & \text{if } x-x_0<(\varDelta_0+6\varLambda)\tau\\ (x-x_0-\varDelta_0\tau)/6\tau & \text{if } (\varDelta_0+6\varLambda)\tau<x-x_0<\varDelta_0\tau\\ 0 & \text{if } x-x_0>\varDelta_0\tau.\end{cases} \tag{2.35}$$

The rarefaction occupies the expansion fan

$$\varDelta_0+6\varLambda\leqslant\frac{x-x_0}{\tau}\leqslant\varDelta_0. \tag{2.36}$$

Now in the limit $\varDelta_1\to 0$ the rarefaction can only fully propagate to within the vicinity of the contraction if $6\varLambda\geqslant -\varDelta_0$. Otherwise, and, in particular if $\varDelta_0<0$, there will be no interaction. If $-6\varLambda\gg\varDelta_0>0$ then it would again be expected that the interaction between the rarefaction and contraction will be rapid and the rarefaction will propagate upstream leaving a subcritical solution in the vicinity of the contraction.

3. Unsteady hydraulic theory

For long contractions the solution technique of Grimshaw & Smyth (1986) and Clarke & Grimshaw (1994) suggest that we consider the solutions of (2.25), (2.26) and (2.27) in the hydraulic limit. In this limit (2.25) reduces to the characteristic equation

$$\frac{\mathrm{d}}{\mathrm{d}x}(\varDelta A+3A^2)=0\quad\text{on}\ \frac{\mathrm{d}x}{\mathrm{d}\tau}=\varDelta+6A, \tag{3.1}$$

which can be integrated to give the solution

$$6A=-\varDelta\pm(\varDelta^2+12f_0)^{1/2}\quad\text{on}\ \frac{\mathrm{d}x}{\mathrm{d}\tau}=\pm(\varDelta^2+12f_0)^{1/2}, \tag{3.2a}$$

where

$$f_0 = 3\Lambda^2 S(x_0)^2 + \Delta(x_0)\Lambda S(x_0), \tag{3.2b}$$

and $x_0 = x(0)$ is the initial position of each characteristic. The choice of the sign in this solution is initially that of $\Delta(x_0) + 6\Lambda S(x_0)$, and then it reverses if the characteristic has a turning point, i.e. $\mathrm{d}x/\mathrm{d}\tau = 0$. It is the characteristics with turning points which are of particular interest here, as these correspond to the weakly nonlinear limit of hydraulically controlled solutions.

Consider first the case $\Lambda > 0$, which in the hydraulic limit can be called a compressive front. Initially the front is located downstream of the contraction. In the hydraulic limit the compressive front will form a shock, and then, as described in the previous section, dispersive effects must be invoked, and the front develops into an undular bore. From the results obtained in the previous section, we infer that the undular bore will only reach the contraction if $4\Lambda > -\Delta_0$, and that if $6\Lambda < \Delta_0$ there will be only a rapid interaction with the contraction. The remaining cases must be studied numerically, and this is taken up in § 5.

Next, suppose that $\Lambda < 0$ which is the rarefaction front of the previous section. We infer that only for this case can hydraulic control occur in the contraction, and so we consider this now in some detail. From the results of the previous section, we infer that the rarefaction cannot reach the contraction if $\Delta_0 \leqslant -6\Lambda$, so henceforth we let $\Delta_0 > -6\Lambda$, so that in particular $\Delta_0 > 0$ (i.e. the upstream velocity perturbation is subcritical). Then three regimes can be identified. Using the parameter λ, (2.30), we see that $\lambda\Delta_0 > 1$. Then, in the first regime

$$0 < \lambda\Delta_1 < \lambda\Delta_0 - (2\lambda\Delta_0 - 1)^{1/2}. \tag{3.3}$$

Here no turning points occur in the contraction or downstream, and so the front is able to propagate through the contraction, leaving a steady, subcritical solution in its wake. For long times, all the characteristics in the contraction emanate from downstream, where

$$f_0 = 3\Lambda^2 + \Delta_0\Lambda, \tag{3.4}$$

and hence the steady solution is

$$6A = -\Delta + (\Delta^2 + 12\Delta_0\Lambda + 36\Lambda^2)^{1/2}. \tag{3.5}$$

Outside the contraction, $A = \Lambda$, and upstream this steady solution is preceded by the rarefaction front, with the form (2.35). In the second regime, again $\Delta_0 > \Delta_1$, but now

$$\lambda\Delta_0 - (2\lambda\Delta_0 - 1)^{1/2} < \lambda\Delta_1 < \lambda\Delta_0, \tag{3.6}$$

so that the characteristics have turning points in the contraction, although the minimum value of Δ, $\Delta_0 - \Delta_1$, remains positive. The characteristics which lead to a steady solution are those which have an asymptotic turning point at the throat of the contraction so that,

$$12 f_0 = -(\Delta_0 - \Delta_1)^2. \tag{3.7}$$

Thus from (3.2), asymptotically, a steady hydraulically controlled solution will form in the vicinity of the contraction of the form

$$6A = -\Delta + \operatorname{sgn} x(\Delta^2 - (\Delta_0 - \Delta_1)^2)^{1/2}. \tag{3.8}$$

In the hydraulic theory, this corresponds to a normal control as the solution is controlled at the maximum constriction. Upstream of the contraction this is again preceded by a rarefaction of amplitude A_+, where A_+ is the upstream limiting value of

A in (3.8). Downstream a shock must form as the characteristics with turning points in the contraction carrying the value A_-, where A_- is the downstream limiting value of A in (3.8), will intersect with characteristics emanating from downstream carrying the value Λ, and $A_- < \Lambda$. This shock will have the asymptotic velocity

$$V = 3\Lambda + \tfrac{1}{2}\Delta_0 - \tfrac{1}{2}(\Delta_0^2 - (\Delta_0 - \Delta_1)^2)^{1/2}. \tag{3.9}$$

Of course, this shock can be resolved by dispersive effects to have the structure of an undular bore. In the final regime, where

$$\Delta_1 \geqslant \Delta_0, \tag{3.10}$$

then at some point $x_c \leqslant 0$, $\Delta(x_c) = 0$. The characteristics which lead to a steady solution now emanate from the base of the rarefaction where $A = 0$ and $f_0 = 0$. Now, the resulting steady solution is

$$6A = \begin{cases} 0 & \text{if } x > x_c \\ -2\Delta & \text{if } x < x_c. \end{cases} \tag{3.11}$$

This solution, where the control occurs away from the maximum topographic perturbation, corresponds to a virtual control. Again, downstream of the contraction there is a shock, which will have the asymptotic velocity 3Λ.

4. Rarefaction fronts ($\Lambda < 0$)

Consider solutions of the full system (2.25), (2.26) and (2.27) where again we assume that $0 > 6\Lambda \geqslant -\Delta_0$. The hydraulic approximation, although it provides an indication of the behaviour of trapped rarefactions, is obviously deficient in two ways. First, it is only valid for long contractions. Secondly, it is invalid in the vicinity of shocks where rapid changes in amplitude occur. To proceed further dispersion must be reintroduced into the problem. This can be done by constructing an asymptotic approximate solution, or by solving the problem numerically.

4.1. *Asymptotic matched solutions*

The hydraulic solution demonstrated that in the critical regime the asymptotic flow will be steady in the vicinity of the contraction, upstream a rarefaction will propagate away from the contraction and downstream a shock will propagate away from the contraction. Following the solution technique of Smyth (1987) and Clarke & Grimshaw (1994) an asymptotic solution can be constructed which puts detail on these features of the hydraulic solution. In constructing this solution we are not concerned with the transient interaction of the rarefaction with the contraction, rather with the long-time solution that emerges. In the terminology of Clarke & Grimshaw (1994) it would be expected that there is a strong and weak resonant regime.

In the strong resonant regime steady solutions of (2.25) having constant, unequal limits as $x \to \pm\infty$, denoted as $A_\pm$ respectively, are sought. These solutions must satisfy

$$\Delta A + 3A^2 + A_{xx} = \Delta_0 A_\pm + 3A_\pm^2. \tag{4.1}$$

In the limit $|\Lambda|^{1/2}x_a \gg 1$ the hydraulic approximation can again be used, and the relevant solutions are (3.8) if $\Delta > 0$, and (3.11) if $\Delta \leqslant 0$ somewhere in the contraction. In the limit $|\Lambda|^{1/2}x_a \ll 1$ we can approximate θ, (2.27), with the Dirac delta function, $\delta(\cdot)$, so that

$$\Delta = \Delta_0 - \Delta_1 D\delta(x), \tag{4.2a}$$

and

$$D = x_a \int_{-\infty}^{\infty} \theta(x)\,\mathrm{d}x. \tag{4.2b}$$

A solution is sought of the form

$$A = \begin{cases} A_+ & \text{if } x > 0 \\ A_- + 2k^2 \operatorname{sech}^2 k(x - x_0) & \text{if } x < 0, \end{cases} \tag{4.3}$$

and it can be shown that

$$16k^3 - 3^{1/2} \varDelta_1 D(\varDelta_0 - 4k^2) = 0, \tag{4.4a}$$

$$x_0 = \frac{1}{2k} \ln (2 + 3^{1/2}), \tag{4.4b}$$

$$6A_{\pm} = -\varDelta_0 \pm 4k^2. \tag{4.4c}$$

The only branch of solutions for k of relevance for rarefactions is that upon which $k = 0$ when $\varDelta_1 D = 0$. Therefore, for $\varDelta_1 D \geqslant 0$

$$0 \leqslant 4k^2 < \varDelta_0. \tag{4.5}$$

For intermediate length contractions numerical solutions must be used to obtain A_+ and A_-. However, it is sufficient here to note that for both long and short contractions

$$-\varDelta_0 \leqslant 6A_+ \leqslant 0 \quad \text{and} \quad -2\varDelta_0 \leqslant 6A_- \leqslant -\varDelta_0. \tag{4.6}$$

Upstream of the contraction A_+ must be matched to $A = 0$, hence as $A_+ < 0$ a rarefaction will form with the approximate long-time solution

$$A = \begin{cases} A_+ & \text{if } x < (\varDelta_0 + 6A_+)\tau \\ (x - \varDelta_0 \tau)/6\tau & \text{if } (\varDelta_0 + 6A_+)\tau < x < \varDelta_0 \tau \\ 0 & \text{if } x > \varDelta_0 \tau. \end{cases} \tag{4.7}$$

Obviously, when a virtual control occurs and $A_+ = 0$ there is no upstream rarefaction. Downstream of the contraction A_- must be matched to the downstream level of the shear front, $\varLambda$. In the hydraulic approximation, as $\varLambda > A_-$ a shock forms, which is modelled as a modulated cnoidal wavetrain of the form (2.32). As shown by Smyth (1987), the appropriate solution of the characteristic equations for the downstream shock is

$$\left.\begin{aligned} b &= \varLambda + \mu\left(2 - m - \frac{2E}{K}\right) \\ a &= -\mu m \\ \frac{x}{\tau} &= \varDelta_0 + 6\varLambda + 2\mu\left(2 - m + \frac{2m(1-m)K}{E - (1-m)K}\right) \end{aligned}\right\} \text{for } 0 \leqslant m \leqslant 1. \tag{4.8}$$

As $m \to 1$ the modulated wavetrain must match with the outer limit of the inner solution, and so

$$\mu = A_- - \varLambda. \tag{4.9}$$

The modulated wavetrain occupies the expansion fan

$$12(A_- - \varLambda) \leqslant \frac{x}{\tau} - (\varDelta_0 + 6\varLambda) \leqslant 2(A_- - \varLambda), \tag{4.10}$$

consequently, if

$$\Delta_0 + 6\Lambda + 2(A_- - \Lambda) \geqslant 0, \tag{4.11}$$

the modulated wavetrain does not propagate away from the contraction and the strong resonant solution no longer applies.

In the weak resonant regime a lee wavetrain forms downstream of the contraction, which is matched to the downstream limit Λ by a modulated wavetrain. The lee wave must satisfy

$$A = \Lambda + \mu m_s(1 - 2\mathrm{cn}^2\kappa x), \tag{4.12}$$

where $\kappa^2 = -\mu$ and m_s is the modulus of the lee wave. The amplitude will still be steady in this region, hence (4.1) will apply, giving the two conditions

$$\mu^2 m_s^2 = \Delta_0 A_+ + 3A_+^2 - \Delta_0 \Lambda - 3\Lambda^2, \tag{4.13a}$$

$$2\mu(2 - m_s) = -(\Delta_0 + 6\Lambda). \tag{4.13b}$$

The second of these is equivalent to setting the cnoidal wavetrain velocity to zero in (2.32) and (4.8). If A_+ is known then μ and m_s can be found. This is the case for long contractions where the upstream solution must remain unchanged from (3.8) and (3.11) as the flow is controlled in the contraction and any wave which can propagate upstream will be transient and unsteady. Hence, dispersion can still be ignored in this region. For short contractions this is not so, as through the jump conditions A_+ is dependent on the downstream solution. Here a solution is sought of the form

$$A = \begin{cases} A_+ & \text{if } x > 0 \\ \Lambda + \mu m_s(1 - 2\mathrm{cn}^2\kappa(x - x_0)) & \text{if } x < 0, \end{cases} \tag{4.14}$$

where $\kappa^2 = -\mu$. Then from the continuity conditions across $x = 0$

$$A_+ = \Lambda + \mu m_s(1 - 2\mathrm{cn}^2\kappa x_0), \tag{4.15a}$$

$$(\Delta_1 D A_+)^2 = -16\mu^3 m_s^2 \mathrm{cn}^2\kappa x_0(1 - \mathrm{cn}^2\kappa x_0)(1 - m_s + m_s\mathrm{cn}^2\kappa x_0). \tag{4.15b}$$

These two conditions can then be used with (4.13) to obtain the steady solution in the contraction. Immediately downstream the lee wave will have the solution (4.12), and will occupy the expansion fan

$$\Delta_0 + 6\Lambda + 2\mu\left(2 - m_s + \frac{2m_s(1 - m_s)K_s}{E_s - (1 - m_s)K_s}\right) \leqslant \frac{x}{\tau} \leqslant 0, \tag{4.16}$$

where $E_s = E(m_s)$ and $K_s = K(m_s)$. The modulated wavetrain again has the solution (4.8), however, m is now in the range $0 \leqslant m \leqslant m_s$, and the wavetrain occupies the expansion fan

$$12\mu \leqslant \frac{x}{\tau} - (\Delta_0 + 6\Lambda) \leqslant 2\mu\left(2 - m_s + \frac{2m_s(1 - m_s)K_s}{E_s - (1 - m_s)K_s}\right). \tag{4.17}$$

The limits for the resonant regimes for the cases of long and short contractions can now be determined. For long contractions the limits of the strong and weak regimes are respectively

$$S: \ \lambda\Delta_1 \geqslant \lambda\Delta_0 - (\lambda^2\Delta_0^2 - 4(\lambda\Delta_0 - 1)^2)^{1/2} \quad \text{and } 1 \leqslant \lambda\Delta_0 \leqslant 2, \tag{4.18a}$$

$$W:\begin{cases}\lambda\Delta_0-(2\lambda\Delta_0-1)^{1/2}\leqslant\lambda\Delta_1 & \\ \leqslant\lambda\Delta_0-(\lambda^2\Delta_0^2-4(\lambda\Delta_0-1)^2)^{1/2} & \text{if } 1\leqslant\lambda\Delta_0\leqslant 2\\ \lambda\Delta_1\geqslant\lambda\Delta_0-(2\lambda\Delta_0-1)^{1/2} & \text{if } \lambda\Delta_0\geqslant 2.\end{cases}\qquad(4.18b)$$

For short contractions there is only a distinct strong resonant regime, given by

$$\lambda^{1/2}D\Delta_1\geqslant 4\left(\frac{2}{3}\right)^{1/2}\frac{(\lambda\Delta_0-1)^{3/2}}{(2-\lambda\Delta_0)}\quad\text{and } 1\leqslant\lambda\Delta_0\leqslant 2.\qquad(4.19)$$

In the subcritical regime lee waves are generated for all values of $\Delta_1 D$.

Assuming a smooth variation between the limits of long and short contractions these results suggest that strong resonant behaviour, where a constant mean level is generated downstream of the contraction, only occurs if the incident rarefaction satisfies

$$-\Delta_0\leqslant 6A\leqslant -\tfrac{1}{2}\Delta_0.\qquad(4.20)$$

4.2. *Numerical solutions*

To solve (2.25) numerically a simple adaption of the method of Fornberg & Whitham (1978) is used, which uses Fourier transforms to evaluate the derivatives. However as problems involving fronts have unequal limits as $x\to\pm\infty$, which we denote here as $A^{\pm}$ respectively, these cannot be used. This can be overcome by writing

$$A=A^{+}+\tfrac{1}{2}(A^{-}-A^{+})(1-\tanh x)+A'(x,t).\qquad(4.21)$$

The derivatives of the first term are evaluated analytically, whereas Fourier transforms are used to evaluated the derivatives of the residue A'. Sponge layers are used at the boundaries of the domain to avoid problems due to periodic boundary conditions. Apart from this the method is essentially the same as that of Fornberg & Whitham (1978).

Two examples of the numerical solutions are shown in figure 1, the first from the strong resonant regime and the second from the weak resonant regime. Both clearly display the features outlined in the asymptotic matched solutions of the previous subsection. In figure 1(*a*) the strong resonant example is shown. The rarefaction propagates towards the contraction, and the result of this interaction is that a steady solution forms in the contraction, a rarefaction propagates upstream and a modulated wavetrain propagates downstream. In figure 1(*b*) an example of the weak resonant regime is shown. Now, a lee wave forms downstream of the contraction instead of the modulated wavetrain.

5. Undular bores ($A>0$)

Consider now solutions of (2.25), (2.26) and (2.27) for

$$4A\geqslant\max(-\Delta_0,0).\qquad(5.1)$$

In this compressive case the hydraulic limit implies that a shock forms before the front reaches the contraction, and so dispersion must then be included, with the outcome being the formation of an undular bore. Consequently, to investigate the interaction of undular bores with contractions, we shall resort to numerical solutions.

5.1. *Long contractions*

In the limit $A^{1/2}x_a\gg 1$ the boundaries of any regimes which occur will be a function of $\lambda\Delta_0$ and $\lambda\Delta_1$. The classifications of numerical solutions on this basis are shown

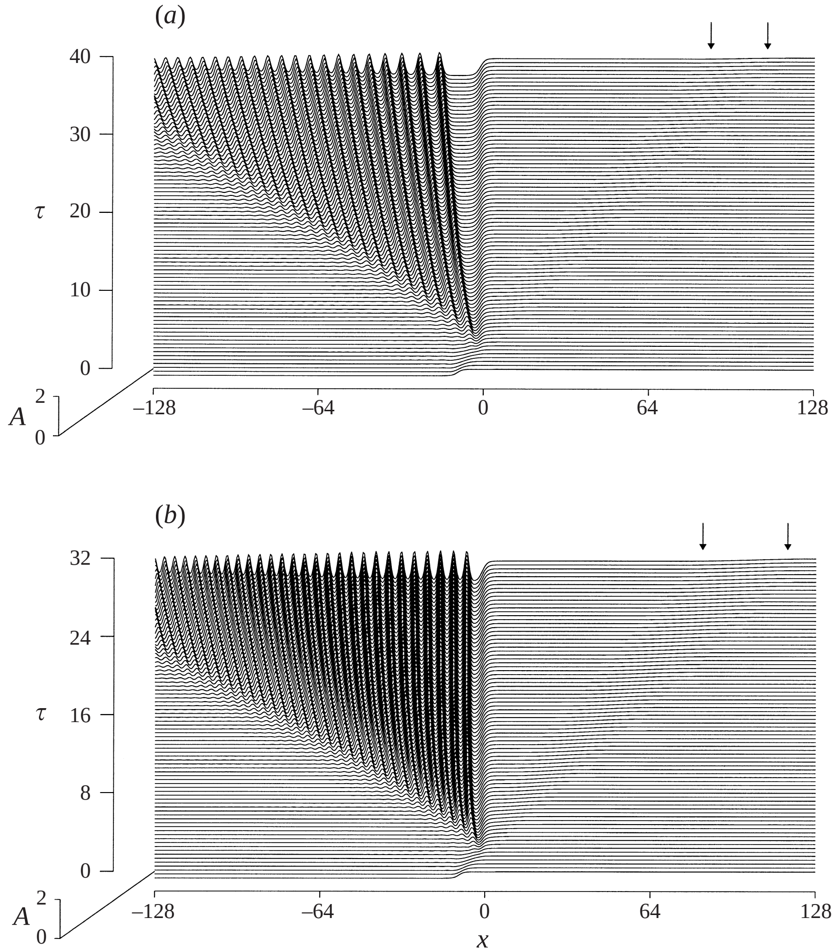

FIGURE 1. Numerical solutions of (2.25), (2.26) and (2.27) for rarefactions with $\Lambda = -1/3$, $x_a = 2$ and $\Delta_1 = 2$. In (*a*) a strong resonant solution with $\Delta_0 = 3$ and in (*b*) a weak resonant solution for $\Delta_0 = 4$. The approximate final limits of the upstream propagating rarefaction is shown in both parts by vertical arrows. In this and subsequent figures the imposed flow is from right to left and the contraction is centred at $x = 0$.

in figure 2. Three types of solutions occur, strong and weak resonant flows and subcritical flows. The empirical limits of the strong and weak resonant regimes are found to be

$$S : 4\lambda\Delta_1 \geqslant 9(\lambda\Delta_0 + 1)^2 - 1 \quad \text{if } \lambda\Delta_0 \geqslant -\tfrac{2}{3}, \tag{5.2a}$$

$$W : \begin{cases} 4\lambda\Delta_1 \leqslant 9(\lambda\Delta_0 + 1)^2 - 1 & \text{if } -\tfrac{2}{3} \leqslant \lambda\Delta_0 \leqslant \tfrac{1}{2} \\ 4\lambda\Delta_1 \leqslant 9(\lambda\Delta_0 + 1)^2 - 1, \quad \lambda|\Delta_0 - \Delta_1| \geqslant (2\lambda\Delta_0 - 1)^{1/2} & \text{if } \lambda\Delta_0 \geqslant \tfrac{1}{2}. \end{cases} \tag{5.2b}$$

First, a strong resonant regime occurs adjacent to the line $4\Lambda = -\Delta_0$. A typical solution from this regime is shown in figure 3(*a*). Here the behaviour can be characterized as individual solitary waves of the undular bore interacting with the contraction, rather than the undular bore as a whole. Initially the bulk of the expansion fan propagates downstream, however the leading wave of the undular bore eventually propagates toward the contraction. In the contraction the solitary wave is amplified,

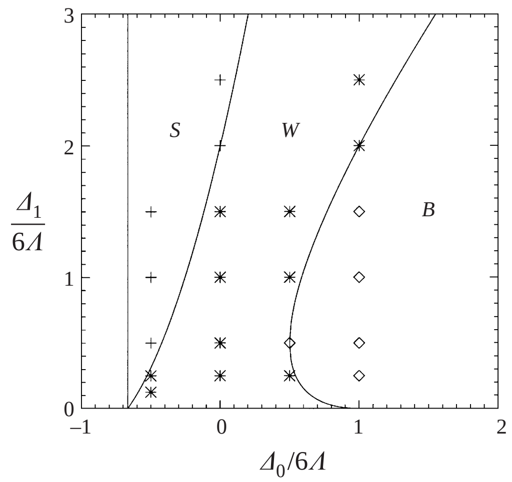

FIGURE 2. The classification of numerical solutions of (2.25), (2.26) and (2.27) for $\Delta > 0$ and $\Delta^{1/2}x_a \gg 1$. Three distinct regimes are found, the strong resonant regime (S), the weak resonant regime (W) and the subcritical regime (B).

decays and eventually propagates upstream. As part of this interaction process a small amplitude wave is sent downstream from the throat of the contraction, which interacts with the bulk of the undular bore which remains downstream. Eventually a second solitary wave can separate off from the undular bore and propagate upstream, repeating the interaction process. Further numerical simulations suggest that this process is repeated indefinitely, with the bulk of the undular bore being positioned a finite distance downstream from the throat of the contraction and solitary waves periodically separating off and propagating upstream through the contraction.

In the weak regime the solutions are characterized by a continuous interaction of the undular bore with the contraction, an example of this being shown in figure 3(b). The limits of the expansion fan can be clearly seen to be propagating upstream and downstream in this case. Waves with modulus greater than some value m_0 can now propagate upstream through the contraction. In the contraction the waves are amplified, decay and then propagate upstream. An unsteady interaction process continues indefinitely in the contraction with waves of modulus m_0 propagating through the contraction and upstream.

In the final regime rapid interactions occur between the undular bore and the contraction with the whole undular bore propagating upstream leaving a subcritical solution of the form (3.5) in the vicinity of the contraction. An example of this is shown in figure 3(c).

5.2. *Short contractions*

In the limit $\Delta^{1/2}x_a \ll 1$ the regimes for the types of solutions should be functions of $\lambda\Delta_0$ and $\lambda^{1/2}D\Delta_1$. A classification of a range of numerical solutions based on this scaling is shown in figure 4. Again, three distinct regimes occur, however the behaviour differs from that for long contractions. Here the empirical limits of the strong and unsteady weak resonant regimes are found to be

$$S : \lambda^{1/2}D\Delta_1 \geqslant 3\lambda\Delta_0 + 2 \quad \text{if } \lambda\Delta_0 \geqslant -\tfrac{2}{3}, \tag{5.3a}$$

$$W_u : \begin{cases} \lambda^{1/2}D\Delta_1 \leqslant 3\lambda\Delta_0 + 2 & \text{if } -\tfrac{2}{3} \leqslant \lambda\Delta_0 \leqslant 1 \\ (1-(\lambda\Delta_0)^{-1})^{1/3}(3\lambda\Delta_0+2) \leqslant \lambda^{1/2}D\Delta_1 \leqslant 3\lambda\Delta_0 + 2 & \text{if } \lambda\Delta_0 > 1. \end{cases} \tag{5.3b}$$

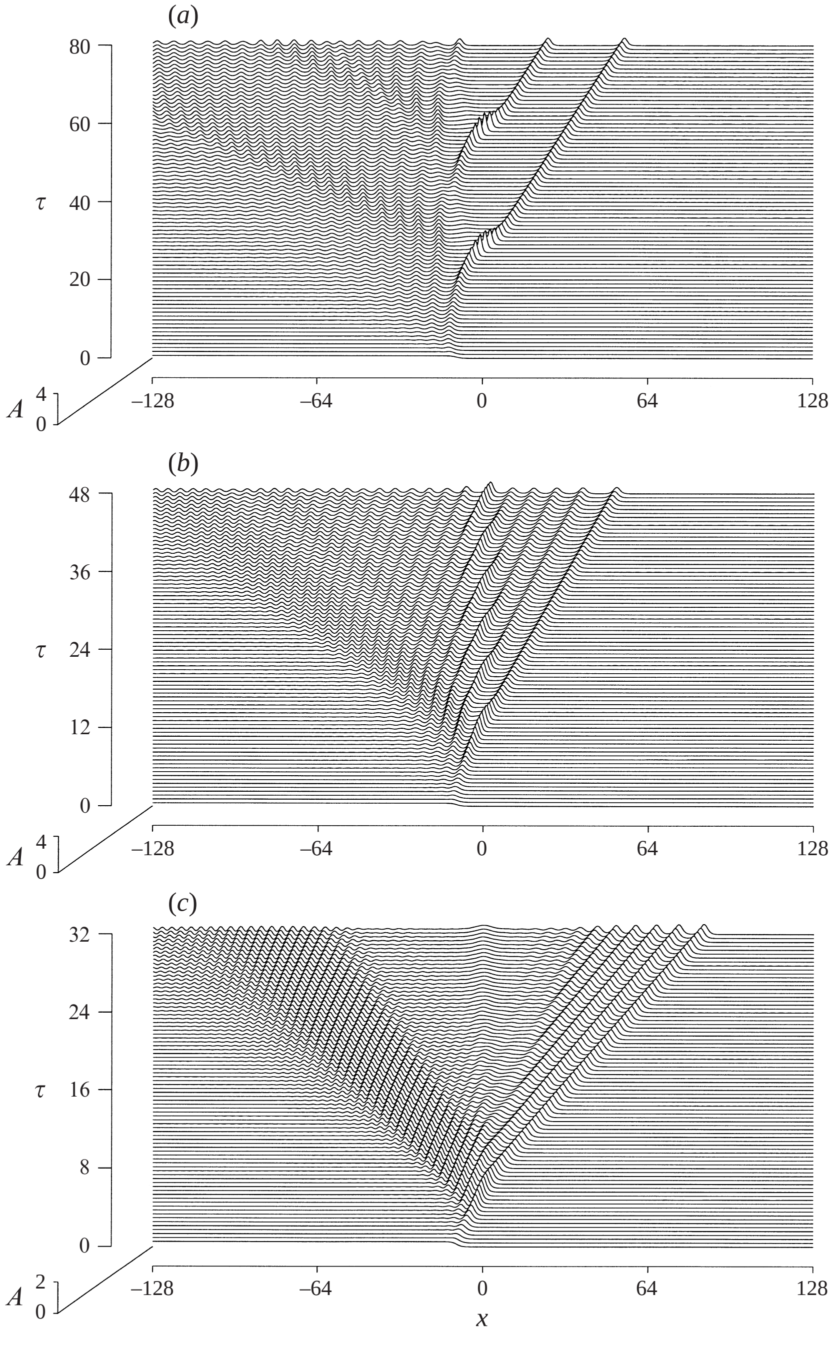

FIGURE 3. Three solutions of (2.25), (2.26) and (2.27) for $\Lambda = 1/3$, $x_a = 5$ and $\Delta_1 = 2$, corresponding to a solution from each of the regimes in figure 2. (*a*) A strong resolution solution where $\Delta_0 = -1$; (*b*) a weak resonant solution where $\Delta = 0$ and in (*c*) a subcritical solution where $\Delta_0 = 2$.

Again a strong resonant regime occurs adjacent to the line $4\Lambda = -\Delta_0$. An example is shown in figure 5(*a*). In contrast to the regime for long contractions, here after the interaction a quasi-steady shelf and a downstream propagating undular bore forms downstream of the contraction. Solitary waves are formed on the downstream side

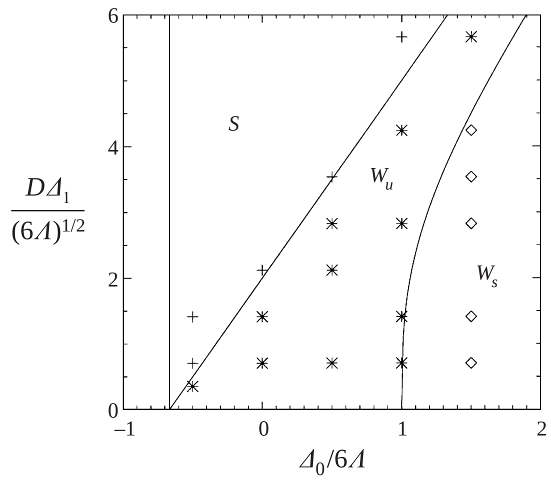

FIGURE 4. The classification of numerical solutions of (2.25), (2.26) and (2.27) for $\Lambda > 0$ and $\Lambda^{1/2}x_a \ll 1$. Three distinct regimes are found, the strong resonant regime (S), the unsteady weak resonant regime (W_u) and the steady weak regime (W_s). This last regime represents the subcritical solutions.

of the contraction and then propagate upstream. For each solitary wave a small-amplitude solitary wave is also generated and propagates downstream on the shelf, interacting with the downstream undular bore. The behaviour in this regime is similar to the corresponding strong regime for rarefactions, except here the solution in the vicinity of the contraction is always unsteady.

In contrast to rarefactions there are now two weak regimes, an unsteady and a steady weak regime. In the unsteady weak regime the undular bore remains trapped in the vicinity of the contraction, an example of which is shown in figure 5(*b*). The characteristic features of the long-time solution are that a quasi-steady lee wavetrain forms downstream of the contraction and large-amplitude solitary waves propagate upstream from the contraction. As in the strong resonant regime these solitary waves are formed immediately downstream of the contraction and then propagate upstream through the contraction. This generation process creates small amplitude waves which propagate downstream, interacting with the lee-waves. Thus in the vicinity of the contraction solutions again remain unsteady for all time.

In the steady weak regime the features are similar except here the upstream undular bore is able to propagate away from the contraction. An example is shown in figure 5(*c*). In consequence immediately upstream of the contraction the amplitude is constant. Immediately downstream a steady lee wavetrain forms. This regime is effectively the subcritical regime for short contractions.

6. Conclusions

A model of the interaction of weakly nonlinear fronts in a stratified fluid with contractions has been considered here in an attempt to understand the dynamics of the establishment of virtual controls. It has been first argued that the appropriate model equation for this process is a variable coefficient KdV equation, (2.25). Solutions of this equation demonstrate that virtual controls will only occur for rarefactions whose amplitudes, Λ, fall in the range (4.20), where Δ_0 is the detuning parameter of the unperturbed flow, and for which the conditions change from subcritical to

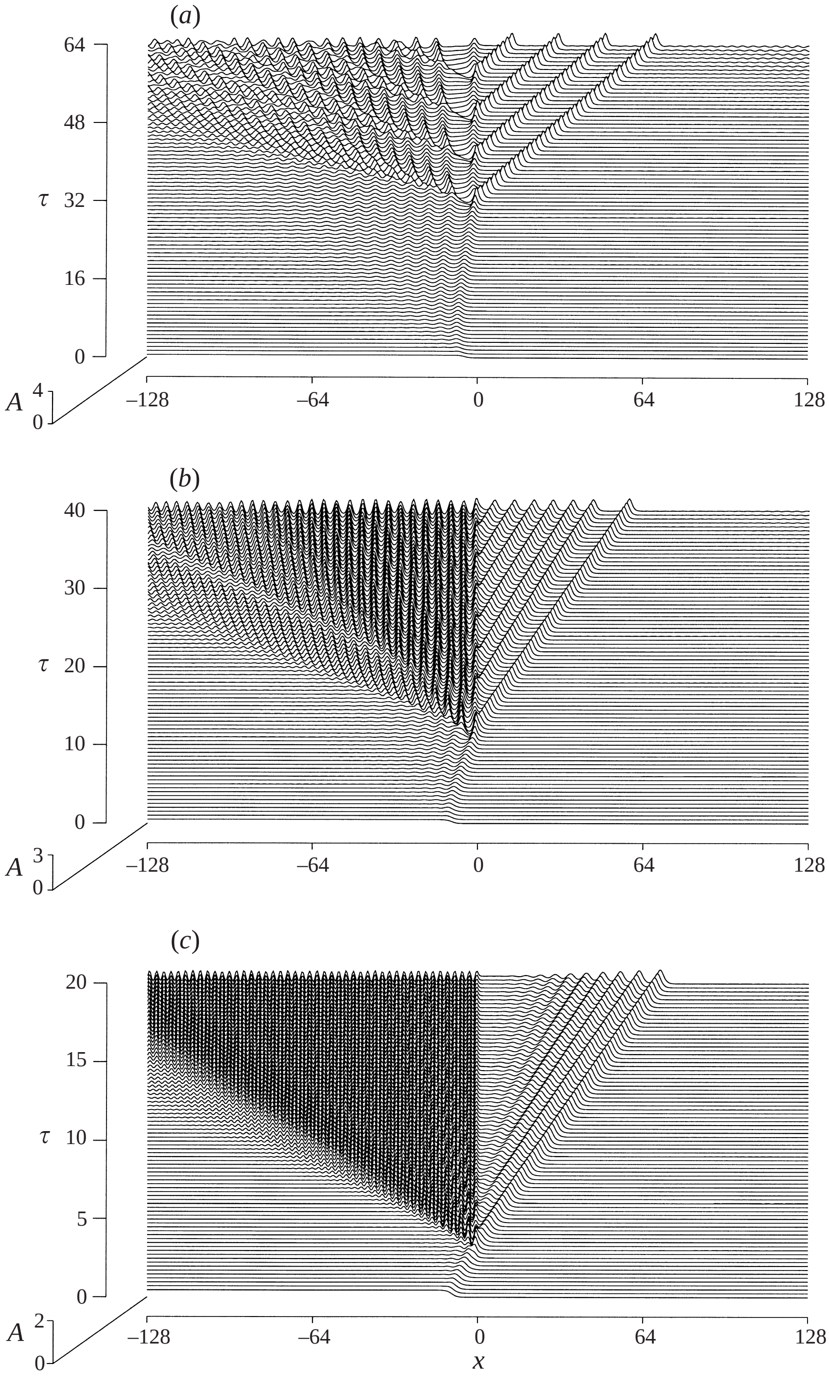

FIGURE 5. Three solutions of (2.25), (2.26) and (2.27) for $\Lambda = 1/3$, $x_a = 1/2$ and $\Delta_1 = 2$, corresponding to a solution from each of the regimes in figure 4. (*a*) A strong resolution solution where $\Delta_0 = -1$; (*b*) a unsteady weak resonant solution where $\Delta = 0$ and in (*c*) a subcritical or steady weak solution where $\Delta_0 = 3$.

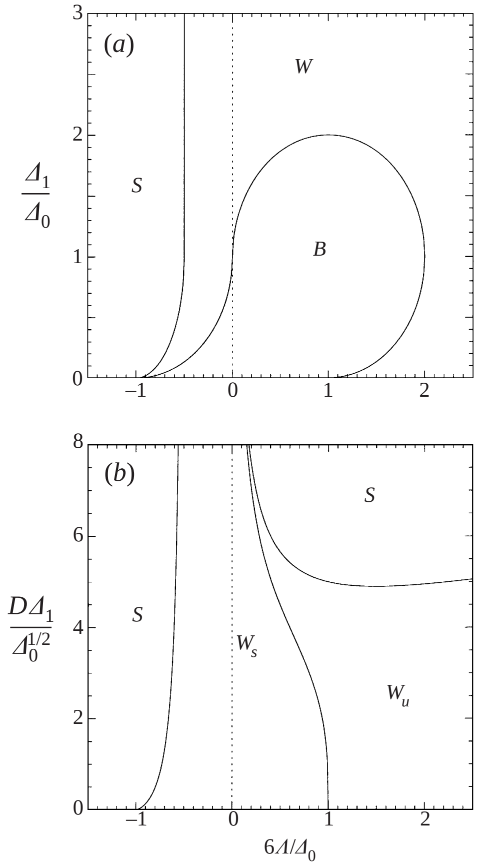

FIGURE 6. The regimes obtained from the theoretical analysis of §4 and the numerical solutions of §5 for Δ_0 positive with Δ_0 used as the scaling parameter rather than Λ. In (*a*) the regimes for long contractions and in (*b*) for short contractions. The notation for the regimes is identical to that used in figures 2 and 4 respectively. Note that in (*a*) the strong resonant regime for positive Λ appears outside of this figure. The locus of the linear solutions of (2.25) ($\Lambda = 0$) is shown on both figures as a dotted line.

supercritical within the contraction. For rarefactions with amplitudes exceeding the upper value of (4.20) a modified form of virtual control is possible for sufficiently long contractions. This incorporates a steady lee wavetrain downstream of the contraction, rather than a constant amplitude shelf as in classical hydraulic theory. These results contrast with the weakly nonlinear limit of unsteady hydraulic theory which predicts virtual controls can occur for rarefactions so long as the conditions change from subcritical to supercritical within the contraction. For undular bores virtual controls cannot occur. Instead, the interaction with contractions is a dynamic process which never settles to a steady state. In the near-critical limit these interactions can take two forms. In the first form part of the undular bore can propagate upstream through the contraction; however, the trailing edge propagates downstream. In the second form, the bulk of the undular bore is trapped downstream of the contraction and periodically the leading wave of the undular bore separates off as a solitary wave and interacts with the contraction.

The behaviour of rarefactions and undular bores has been explicitly considered here, however, there is a third possibility, namely, linear fronts. This can occur in two circumstances, first the trivial case for which $\Lambda \equiv 0$. This case can be dismissed as an exact solution of (2.25) is $\Lambda \equiv 0$. A more important case is that for which the nonlinear coefficient r in (2.23) approaches zero. This occurs for example for a uniformly stratified fluid or a pycnocline located at approximately the mid-depth of a channel. In these circumstances it would be expected that an alternative equation to (2.25) must be derived which balances the leading order dynamical effects of nonlinearity and dispersion and the effect of the variable velocity. However, dispersion may be sufficient to balance the amplification due to the contraction, and so, allow the system to settle down to a steady state. This purely linear problem can be investigated by uniting the results for rarefactions and undular bores, both of which must approach the same limit as $\Lambda \to 0$. The linear problem is only relevant for Δ_0 positive, thus instead of the choice (2.30) we can choose $\lambda = \Delta_0^{-1}$. The various flow regimes can then be replotted using this alternative scaling. In figure 6 this is shown for long and short contractions. The linear solutions now occur on the lines $\Lambda = 0$ of these figures. The solutions break down on this line as the boundary between the subcritical and weak resonant regimes is approached. This must be the case as the weak resonant regime requires the presence of nonlinear effects and so the linear solution becomes secular. For a long contraction the linear solution breaks down exactly where expected when $\Delta_1 = \Delta_0$, i.e. when the conditions change from subcritical to supercritical somewhere in the contraction. However, for a short contraction there is no boundary between the subcritical and weak resonant regimes on the line $\Lambda = 0$, consequently the purely linear solution always remains valid. Between these two limits there must be a critical value of Δ_1 for each value of x_a at which the linear solution breaks down, and it is only as Δ_1 approaches this critical value that an alternative equation to (2.25) must be derived.

The limits of the various near-critical regimes described here are given by (4.18), (4.19), (5.2) and (5.3). However, these expressions are derived in terms of the rescaled vKdV equation (2.25), and so here we reinterpret these results in terms of the original variables. Thus, let ζ, z, x be the isopycnal displacement, the vertical and along-channel coordinate all scaled by the undisturbed fluid depth h, and b the relative width of the channel where $b_{\max} = 1$ far upstream and downstream. Far downstream assume that the incident shear front is $\zeta = \varphi(z)$, where φ is an eigenvector of (2.16) with corresponding eigenvalue c. Then the effective amplitude of the shear front is

$$\Lambda = \frac{3\int_0^1 \varphi_z^3 \, \mathrm{d}z}{2\int_0^1 \varphi_z^2 \, \mathrm{d}z}, \tag{6.1}$$

and so, is a rarefaction front for $\Lambda < 0$ and an undular bore for $\Lambda > 0$. The effect of dispersion is characterized by the parameter μ, where

$$\mu = \frac{\int_0^1 \varphi^2 \, \mathrm{d}z}{2\int_0^1 \varphi_z^2 \, \mathrm{d}z}. \tag{6.2}$$

The Froude number is

$$F = \left(\frac{\bar{u}}{c}\right)^2, \tag{6.3}$$

where $\bar{u}$ is the oncoming flow into the contraction. Then (4.18), (4.19), (5.2) and (5.3)

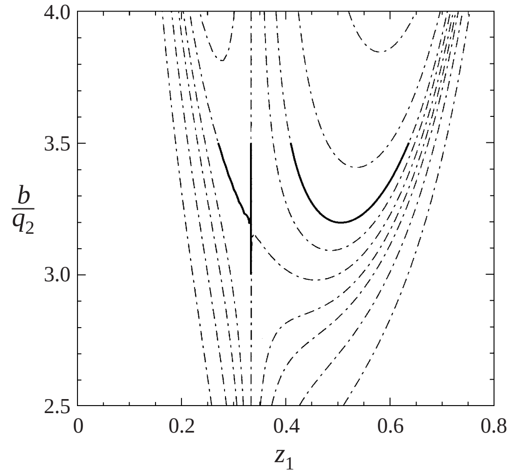

FIGURE 7. A plot of contours of the Bernoulli function for a two-layer fluid, (6.5), for $q_r = 0.5$. Two hydraulically controlled contours are shown in bold. On the right the flow experiences a normal control and on the left a virtual control.

now apply with

$$\lambda \Delta_0 = |\Lambda|^{-1}(1 - F^{1/2}), \tag{6.4a}$$

$$\lambda \Delta_1 = |\Lambda|^{-1}(1 - b_{\min}), \tag{6.4b}$$

$$\lambda^{1/2} D \Delta_1 = (\mu|\Lambda|)^{-1/2} \int_{-\infty}^{\infty} (1 - b)\,\mathrm{d}x. \tag{6.4c}$$

Next, let us consider the results presented here in relation to the two-layer hydraulics of Armi (1986). For a two-layer flow define z_1 and z_2 to be respectively the thicknesses of the upper and lower layers, such that $z_1 + z_2 = 1$. Let q_1 and q_2 be the mass flux in each layer and $q_r = q_1/q_2$. Then, rewriting equation (29) of Armi (1986), in the steady hydraulic limit the interface must correspond to constant values of the Bernoulli function

$$B = z_2 + \frac{q_2^2(z_1^2 - q_r^2 z_2^2)}{2b^2 z_1^2 z_2^2}. \tag{6.5}$$

For $q_r = 0.5$, which corresponds to figure 3(*b*) of Armi (1986), selected contours of B are shown in figure 7. Consider a flow which initially has $z_1 = 1/2$, then the initial velocity is vertically sheared, being twice as strong in the lower layer than in the upper layer. In the near-critical limit there will be a resonant response of the interface and its evolution will be governed by (2.13). Consequently only normal controls can occur at the throat of the contraction, an example of a possible final state is shown in bold in figure 7. The quadratic form of this curve reflects the fact that the amplitude of the interface scales as $O((1 - b_{\min})^{1/2})$. However, if $z_1 = 1/3$ the initial flow through the contraction is uniform. Then the near critical evolution of any waves will be governed by (2.23). In this configuration a rarefaction raises the interface, while an undular bore lowers the interface. If no fronts are generated then a valid solution is $z_1 \equiv 1/3$. Thus the interface achieves a steady state which is flat throughout the contraction, and this steady state may possibly pass through virtual controls on the upstream and downstream side of the throat of the contraction. If a rarefaction of sufficiently large amplitude can propagate to the vicinity of the contraction then at the downstream virtual control it can become trapped and form a solution such as shown in bold. For rarefactions which do not satisfy (4.18), steady lee-waves form downstream of the contraction. If an undular bore propagates to within the vicinity of the contraction then the present theory demonstrates that no steady hydraulic solution can form.

It is rather more problematic to compare the present theory with the hydraulic studies of Wood (1968) and Armi & Williams (1993) of the withdrawal of a continuously stratified fluid from an infinite reservoir. Here, as any steady state can only be set up by fronts propagating upstream from the withdrawal outlets to the vicinity of the contraction, one must consider the ratio of the downstream width of the channel to the minimum width of the channel. In general this ratio is order unity, which creates two problems. First, a large number of modes then have zero velocity on the downstream side of the throat of the contraction, thus one cannot separate out a unique near-critical mode. Secondly, the response of any near-critical front to the velocity changes in the contraction will be $O(1 - b_{\min})$, and so, finite-amplitude effects must be considered. The final difficulty, related to this, is that for both sets of experiments the initial stratification is approximately linear, thus, in a theory such as presented here one must consider finite-amplitude effects as was done by Grimshaw & Yi (1991) for stratified flow over topography. This approach is currently under investigation and will be presented elsewhere. However, some qualitative comparisons can be made. The first point is that the initial flow throughout the domain must be potential flow. Hence, in the vicinity of the contraction the flow can be assumed to be uniform with height, and so, if the conditions at the throat of the contraction are such that the first mode is near-critical then its evolution will be given by a generalization of (2.23). The trapping and interaction of fronts within the contraction then can create the strong shear and separation of isopycnals from the boundaries observed by both Wood (1968) and Armi & Williams (1993). For the hydraulic solution of Wood (1968) the isopycnals are deformed as the flow accelerates through an infinite number of virtual controls on the upstream side of the contraction. Thus we would infer that the first mode experiences a normal control of the type (3.8) at the throat of the contraction. The upstream virtual controls are trivial controls where the amplitude is zero as the higher mode fronts are not able to propagate past the throat of the contraction. On the downstream side of the contraction conditions then must be supercritical with respect to the first mode, consequently they must also be supercritical with respect to the higher modes. However, the change in the velocity and density structure downstream of the throat of the contraction must cause an alteration to the modal structure and long-wave speeds of the higher mode fronts in a similar manner to that described by Clarke & Imberger (1995). Then due to the slowly varying conditions in the contraction it would be expected that these modified fronts would become trapped at virtual controls on the downstream side of the contraction. For the experiments of Armi & Williams (1993) it is apparent that for a number of flows (see their figures 7(*a*), 9, 15(*a*) and 17) the isopycnals are not deformed upstream of the throat of the contraction. Thus we would infer that for these flows the first mode front becomes trapped at some point downstream of the throat of the contraction and experiences a virtual control of the type (3.11).

One special situation for which virtual controls have been postulated to occur, but for which the theory presented here does not apply, is that of exchange flows. Hydraulic theory for this problem for two-layer fluids has been considered by Wood (1970), Armi & Farmer (1986) and Farmer & Armi (1986). These flows have a strong shear and so would seem to contradict the proposition made in Clarke & Grimshaw (1994) that in the weakly nonlinear limit sheared flows would not exhibit virtual controls. However, that proposition is based on the assumption that the near-critical mode is a regular mode. For exchange flows it is clear that the fundamental near-critical mode must be a singular mode with a nonlinear critical layer. Maslowe & Redekopp (1980) and Grimshaw (1981) among others have shown that in the weakly nonlinear, long-wave limit these waves also satisfy the KdV equation. Following this approach it may be possible to derive an equivalent evolution equation to (2.25) which is a model for the establishment of virtual controls in exchange flows. However, as these flows are highly nonlinear it is questionable whether a weakly nonlinear approach can be applied to this problem.

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